SHARP FRACTIONAL HARDY INEQUALITIES IN HALF-SPACES

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Dedicated to V. G. Maz'ya

Abstract. We determine the sharp constant in the Hardy inequality for fractional Sobolev spaces on half-spaces. Our proof relies on a non-linear and non-local version of the ground state representation.

1. Introduction and main results

This short note is motivated by the paper [BD] concerning Hardy inequalities in the half-space $\mathbb{R}^N_+ := \{(x', x_N) :\ x' \in \mathbb{R}^{N-1}, x_N > 0\}$. The fractional Hardy inequality states that for $0 < s < 1$ and $1 \leq p < \infty$ with $ps \neq 1$ there is a positive constant $\mathcal{D}_{N,p,s}$ such that

$$\iint_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \geq \mathcal{D}_{N,p,s} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}}\, dx \tag{1.1}$$

for all $u \in C_0^\infty(\overline{\mathbb{R}^N_+})$ if $ps < 1$ and for all $u \in C_0^\infty(\mathbb{R}^N_+)$ if $ps > 1$. In [BD] the sharp (that is, the largest possible) value of the constant $\mathcal{D}_{N,2,s}$ for $p = 2$ is calculated. Our goal in this paper is to determine the sharp constant $\mathcal{D}_{N,p,s}$ for *arbitrary* p.

Indeed, we shall see that the sharp inequality (1.1) follows by a minor modification of the approach introduced in [FS]. In that paper we calculated the sharp constant $\mathcal{C}_{N,p,s}$ in the inequality

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \geq \mathcal{C}_{N,p,s} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}}\, dx \tag{1.2}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$ if $1 \leq p < N/s$ and for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ if $p > N/s$. A (non-sharp) version of (1.2) was used by Maz'ya and Shaposhnikova [MS] in order to simplify and extend considerably a result of Bourgain, Brezis and Mironescu [BBM] on the norm of the embedding $\dot{W}^s_p(\mathbb{R}^N) \subset L_{Np/(N-ps)}(\mathbb{R}^N)$. Our proof of (1.2) relied on a *ground state substitution*, that is, on writing $u(x) = \omega(x)v(x)$ where $\omega(x) = |x|^{-(N-ps)/p}$ is a solution of the Euler-Lagrange equation corresponding to (1.2). In this note we shall prove (1.1) using that $\omega(x) = x_N^{-(1-ps)/p}$ satisfies the Euler-Lagrange equation corresponding to (1.1).

We refer to [BD, D, KMP] and the references therein for motivations and applications of fractional Hardy inequalities.

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In order to state our main result let $1 \leq p < \infty$ and $0 < s < 1$ with $ps \neq 1$ and denote by $\mathcal{W}_p^s(\mathbb{R}_+^N)$ the completion of $C_0^\infty(\mathbb{R}_+^N)$ with respect to the left side of (1.1). It is a consequence of the Hardy inequality that this completion is a space of functions. Moreover, it is well-known that for $ps < 1$, $\mathcal{W}_p^s(\mathbb{R}_+^N)$ coincides with the completion of $C_0^\infty(\overline{\mathbb{R}_+^N})$.

Theorem 1.1 (**Sharp fractional Hardy inequality**). *Let $N \geq 1$, $1 \leq p < \infty$ and $0 < s < 1$ with $ps \neq 1$. Then for all $u \in \mathcal{W}_p^s(\mathbb{R}_+^N)$,*

$$\iint_{\mathbb{R}_+^N \times \mathbb{R}_+^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \geq \mathcal{D}_{N,p,s} \int_{\mathbb{R}_+^N} \frac{|u(x)|^p}{x_N^{ps}}\, dx \tag{1.3}$$

with

$$\mathcal{D}_{N,p,s} := 2\pi^{(N-1)/2} \frac{\Gamma((1+ps)/2)}{\Gamma((N+ps)/2)} \int_0^1 \left|1 - r^{(ps-1)/p}\right|^p \frac{dr}{(1-r)^{1+ps}}\,. \tag{1.4}$$

The constant $\mathcal{D}_{N,p,s}$ is optimal. If $p = 1$ and $N = 1$, equality holds iff u is proportional to a non-increasing function. If $p > 1$ or if $p = 1$ and $N \geq 2$, the inequality is strict for any function $0 \not\equiv u \in \mathcal{W}_p^s(\mathbb{R}_+^N)$.

For $p \geq 2$, inequality (1.3) holds even with a remainder term.

Theorem 1.2 (**Sharp Hardy inequality with remainder**). *Let $N \geq 1$, $2 \leq p < \infty$ and $0 < s < 1$ with $ps \neq 1$. Then for all $u \in \mathcal{W}_p^s(\mathbb{R}_+^N)$ and $v := x_N^{(1-ps)/p} u$,*

$$\begin{aligned} \iint_{\mathbb{R}_+^N \times \mathbb{R}_+^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy - \mathcal{D}_{N,p,s} \int_{\mathbb{R}_+^N} \frac{|u(x)|^p}{x_N^{ps}}\, dx& \\ \geq c_p \iint_{\mathbb{R}_+^N \times \mathbb{R}_+^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{x_N^{(1-ps)/2}} \frac{dy}{y_N^{(1-ps)/2}}& \end{aligned} \tag{1.5}$$

where $\mathcal{D}_{N,p,s}$ is given by (1.4) and $0 < c_p \leq 1$ is given by

$$c_p := \min_{0<\tau<1/2} \left((1-\tau)^p - \tau^p + p\tau^{p-1} \right)\,. \tag{1.6}$$

If $p = 2$, then (1.5) is an equality with $c_2 = 1$.

We conclude this section by mentioning an open problem concerning fractional Hardy–Sobolev–Maz'ya inequalities. If $p \geq 2$ and $0 < s < 1$ with $1 < ps < N$, is it true that the left side of (1.5) is bounded from below by a positive constant times

$$\left(\int_{\mathbb{R}_+^N} |u|^q\, dx \right)^{p/q}, \qquad q = Np/(N-ps)\,?$$

The analogous estimate for $s = 1$,

$$\int_{\mathbb{R}_+^N} |\nabla u|^p\, dx - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}_+^N} \frac{|u(x)|^p}{x_N^p}\, dx \geq \sigma_{N,p} \left(\int_{\mathbb{R}_+^N} |u|^q\, dx \right)^{p/q}, \quad q = Np/(N-p)\,, \tag{1.7}$$

is due to Maz'ya (for $p = 2$) [M] and Barbatis–Filippas–Tertikas (for $2 < p < N$) [BFT]; see also [BFL] for the sharp value of $\sigma_{3,2}$. The proof of (1.7) is based on the analogue of (1.5),

$$\int_{\mathbb{R}^N_+} |\nabla u|^p \, dx - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^p} \, dx \geq c_p \int_{\mathbb{R}^N_+} |\nabla v|^p x_N^{p-1} \, dx \,, \qquad u = x_N^{(p-1)/p} v \,.$$

2. Proofs

2.1. **General Hardy inequalities.** This subsection is a quick reminder of the results in [FS]. Throughout we fix $N \geq 1$, $p \geq 1$ and an open set $\Omega \subset \mathbb{R}^N$. Let k be a non-negative measurable function on $\Omega \times \Omega$ satisfying $k(x,y) = k(y,x)$ for all $x, y \in \Omega$ and define

$$E[u] := \iint_{\Omega\times\Omega} |u(x) - u(y)|^p k(x,y) \, dx \, dy \ .$$

Our key assumption for proving a Hardy inequality for the functional E is the following.

Assumption 2.1. Let ω be an a.e. positive, measurable function on Ω. There exists a family of measurable functions k_ε, $\varepsilon > 0$, on $\Omega \times \Omega$ satisfying $k_\varepsilon(x,y) = k_\varepsilon(y,x)$, $0 \leq k_\varepsilon(x,y) \leq k(x,y)$ and

$$\lim_{\varepsilon\to 0} k_\varepsilon(x,y) = k(x,y) \tag{2.1}$$

for a.e. $x, y \in \Omega$. Moreover, the integrals

$$V_\varepsilon(x) := 2 \ \omega(x)^{-p+1} \int_\Omega (\omega(x) - \omega(y)) \, |\omega(x) - \omega(y)|^{p-2} \, k_\varepsilon(x,y) \, dy \tag{2.2}$$

are absolutely convergent for a.e. x, belong to $L_{1,\mathrm{loc}}(\Omega)$ and $V := \lim_{\varepsilon\to 0} V_\varepsilon$ exists weakly in $L_{1,\mathrm{loc}}(\Omega)$, i.e., $\int V_\varepsilon g \, dx \to \int V g \, dx$ for any bounded g with compact support in Ω.

The following abstract Hardy inequality was proved in [FS] in the special case $\Omega = \mathbb{R}^N$. The general case considered here is proved by exactly the same arguments.

Proposition 2.2. *Under Assumption 2.1, for any u with compact support in Ω and $E[u]$ and $\int V_+ |u|^p \, dx$ finite one has*

$$E[u] \geq \int_\Omega V(x) |u(x)|^p \, dx \ . \tag{2.3}$$

For $p \geq 2$, a stronger version of (2.3) is valid which includes a remainder term.

Proposition 2.3. *Let $p \geq 2$. Under Assumption 2.1, for any u with compact support in Ω write $u = \omega v$ and assume that $E[u]$, $\int V_+ |u|^p \, dx$, and*

$$E_\omega[v] := \iint_{\Omega\times\Omega} |v(x) - v(y)|^p \, \omega(x)^{\frac{p}{2}} k(x,y) \omega(x)^{\frac{p}{2}} \, dx \, dy$$

are finite. Then

$$E[u] - \int_\Omega V(x)|u(x)|^p \, dx \geq c_p \, E_\omega[v] \tag{2.4}$$

with c_p from (1.6). *If $p = 2$, then* (2.4) *is an equality with $c_2 = 1$.*

2.2. **Proof of Theorem 1.1.** Throughout this subsection we fix $N \geq 1$, $0 < s < 1$ and $p \neq 1/s$ and we abbreviate

$$\alpha := (1 - ps)/p \, .$$

We will deduce the sharp Hardy inequality (1.3) using the general approach in the previous subsection with the choice

$$\omega(x) = x_N^{-\alpha} \, , \quad k(x, y) = |x - y|^{-N-ps} \, , \quad V(x) = \mathcal{D}_{N,p,s} x_N^{-ps} \, . \tag{2.5}$$

The key observation is

Lemma 2.4. *One has uniformly for x from compacts in $\mathbb{R}^N_+$*

$$2 \lim_{\varepsilon \to 0} \int_{y \in \mathbb{R}^N_+ \, , |x_N - y_N| > \varepsilon} (\omega(x_N) - \omega(y_N)) \, |\omega(x_N) - \omega(y_N)|^{p-2} \, k(x, y) \, dy = \frac{\mathcal{D}_{N,p,s}}{x_N^{ps}} \, \omega(x)^{p-1} \tag{2.6}$$

with $\mathcal{D}_{N,p,s}$ from (1.4).

Proof. First, let $N = 1$. Then it follows from [FS, Lem. 3.1] that

$$2 \lim_{\varepsilon \to 0} \int_{y > 0 \, , |x - y| > \varepsilon} (\omega(x) - \omega(y)) \, |\omega(x) - \omega(y)|^{p-2} \, k(x, y) \, dy = \frac{\mathcal{D}_{1,p,s}}{x^{ps}} \, \omega(x)^{p-1}$$

uniformly for x from compacts in $(0, \infty)$. To be more precise, in [FS, Lem. 3.1] the y-integral was extended over the whole axis. Therefore the difference between the constant $\mathcal{C}_{1,s,p}$ in [FS, (3.2)] and our $\mathcal{D}_{1,p,s}$ here comes from the absolutely convergent integral

$$2 \int_{-\infty}^0 (\omega(x) - \omega(|y|)) \, |\omega(x) - \omega(|y|)|^{p-2} \, \frac{dy}{(x - y)^{1+ps}} \, .$$

This proves the assertion for $N = 1$. In order to extend the assertion to higher dimensions we use the fact (see [AS, (6.2.1)]) that

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \frac{dy'}{(|x' - y'|^2 + m^2)^{(N+ps)/2}} &= |\mathbb{S}^{N-2}| m^{-1-ps} \int_0^\infty \frac{r^{N-2} \, dr}{(r^2 + 1)^{(N+ps)/2}} \\ &= \frac{1}{2} |\mathbb{S}^{N-2}| m^{-1-ps} \frac{\Gamma((N-1)/2) \; \Gamma((1 + ps)/2)}{\Gamma((N + ps)/2)} \end{aligned} \tag{2.7}$$

for $N \geq 2$. Recalling $|\mathbb{S}^{N-2}| = 2\pi^{(N-1)/2} / \Gamma((N-1)/2)$ concludes the proof. $\square$

Proof of Theorem 1.1. According to Lemma 2.4, Assumption 2.1 is satisfied with kernel $k_\varepsilon(x, y) = |x - y|^{-N-ps} \chi_{\{|x_N - y_N| > \varepsilon\}}$. Hence inequality (1.3) for $u \in C_0^\infty(\mathbb{R}^N_+)$ follows from Proposition 2.2. By density it holds for all $u \in \mathcal{W}^s_p(\mathbb{R}^N_+)$. Strictness for $p > 1$ follows by the same argument as in [FS]. In order to discuss equality in (1.3) for $p = 1$ we first note that for equality it is necessary that u is proportional to a non-negative function, which we assume henceforth. From [FS, (2.18)] we see that equality holds iff for a.e. x and y with $\omega(x_N) > \omega(y_N)$ (that is, $x_N < y_N$) one has

$$|\omega(x_N)v(x) - \omega(y_N)v(y)| - (\omega(x_N)v(x) - \omega(y_N)v(y)) = 0$$

for $v(x) := \omega(x_N)^{-1}u(x)$. Since for numbers $a, b \geq 0$ the equality $|a-b| - (a-b) = 0$ holds iff $b \leq a$, we conclude that for a.e. x and y with $x_N < y_N$ one has $\omega(y_N)v(y) \leq \omega(x_N)v(x)$, that is $u(y) \leq u(x)$. If $N = 1$ this means that u is non-increasing. If $N \geq 2$ one sees that for a function u with this property the integral $\int_{\mathbb{R}^N_+} |u| x_N^{-s}\, dx$ is infinite, unless $u \equiv 0$. This proves the strictness assertion in Theorem 1.1.

The fact that the constant is sharp for $N = 1$ was shown in [FS] (with $\mathbb{R}_+$ replaced by $\mathbb{R}$, but this only leads to trivial modifications). In order to prove sharpness in higher dimensions we consider functions of the form $u_n(x) = \chi_n(x')\varphi(x_N)$, where

$$\chi_n(x') = \begin{cases} 1 & \text{if } |x'| \leq n\,, \\ n+1-|x'| & \text{if } n < |x'| < n+1\,, \\ 0 & \text{if } |x'| \geq n+1\,. \end{cases}$$

An easy calculation using (2.7) shows that

$$\frac{\iint_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\, dx\, dy}{\int_{\mathbb{R}^N_+} \frac{|u_n(x)|^p}{x_N^{ps}}\, dx} \to A\, \frac{\iint_{\mathbb{R}_+ \times \mathbb{R}_+} \frac{|\varphi(x_N)-\varphi(y_N)|^p}{|x_N-y_N|^{1+ps}}\, dx_N\, dy_N}{\int_{\mathbb{R}_+} \frac{|\varphi(x)|^p}{x_N^{ps}}\, dx_N}$$

as $n \to \infty$ with $A := \frac{1}{2}|\mathbb{S}^{N-2}|\Gamma((N-1)/2)\ \Gamma((1+ps)/2)/\Gamma((N+ps)/2)$. Since $A = \mathcal{D}_{N,p,s}/\mathcal{D}_{1,p,s}$, sharpness of $\mathcal{D}_{N,p,s}$ for $N \geq 2$ follows from sharpness of $\mathcal{D}_{1,p,s}$ for $N = 1$. ☐

Proof of Theorem 1.2. Inequality (1.2) follows immediately from Proposition 2.3. ☐

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